Hooks and Arrows

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Abstract

In this elementary combinatorial note we use a construction due to Mark Haiman to prove the following observation: for any Young diagram the number of hooks of a given shape in the diagram is one less then the number of hooks of the same shape in the complement to the diagram. We further use this fact to provide simple proofs to several previously known results: a bijection which appeared in [3] and certain formulas from [5]. All those results were previously proved in a rather technical way.

1 Inside the Positive Quadrant.

Let D be a Young diagram. We think of D as a subset in $(\mathbb{Z}_{\geq 0})^2$, with the southwest corner box being $(0,0)$. Let $\overline{D} := (\mathbb{Z}_{\geq 0})^2 \backslash D$ be the complement to D in the positive quadrant, and $\widehat{D} := \mathbb{Z}^2 \backslash \overline{D}$ be the complement to $\overline{D}$ in the whole plane $\mathbb{Z}^2$ (i.e. $\widehat{D} = D \sqcup \{(c,d) \in \mathbb{Z}^2 | c < 0 \text{ or } d < 0\}$).

Consider the set of arrows $\{(a,b) \to (c,d) | (a,b) \in \overline{D} \text{ and } (c,d) \in \widehat{D}\}$ pointing from a box in $\overline{D}$ to a box in $\widehat{D}$. If two arrows in A differ by a translation by 1 in vertical or horizontal directions, we say that they are equivalent. This generates an equivalence relation on A. We say that an arrow is *escaping* if it is equivalent to an arrow pointing outside of the positive quadrant.

Note, that there are no north, northeast, or east pointing arrows, and all southwest pointing arrows are escaping. Let $A_{nw} := \{[(a,b) \to (c,d)] \in A | c < a \text{ and } d \geq b\}$ be set of equivalence classes of northwest and west pointing arrows, and $A^0_{nw} \subset A_{nw}$ be the subset of non-escaping classes. The

sets of southeast and south pointing arrows $A^0_{se} \subset A_{se}$ are defined in a similar manner. For the reader convenience, we will refer to arrows from A_{nw} as *northwest arrows*.

There is a natural projection from the set of equivalent classes of arrows to the integer vectors sending an arrow $\mathbf{a} := (a,b) \to (c,d)$ to the vector $\mathbf{v} := (c-a, d-b)$. We say that $\mathbf{a}$ is a $\mathbf{v}$-type arrow in this case.

We will use the following standard definitions of arms and legs of boxes of a diagram:

Definition 1.1. Let $c = (x,y) \in D$ be a box inside a Young diagram D. The *arm* of the box c is the number of boxes inside D in the same row as c, to the east of c. Similarly, the *leg* of the box c is the number of boxes inside D in the same column as c, to the north of c. We use the notations $a(c) = a(x,y)$ and $l(c) = l(x,y)$ for the arm and the leg of the box c correspondingly.

Consider the map $\phi : D \to A^0_{nw}$ defined as follows. Given a box $c = (x,y) \in D$, we define

$$\phi(x,y) = [(x + a(c) + 1, y) \to (x, y + l(c))].$$

The following observation can be found in [4]:

Theorem 1.1. *The map ϕ is a bijection between boxes of the diagram D and equivalence classes of non-escaping northwest arrows A^0_{nw}.*

Proof. Let us move a northwest pointing arrow to the north and to the west as much as possible. If it is not escaping, there will be a unique representative in the class, such that it is impossible to further move it north or west. Suppose that the resulting arrow is $(r,s) \to (k,m)$. Since it is not escaping, we automatically get $r,s,k,m \geq 0$. Since we cannot move it to the north anymore, we have $(k, m+1) \in \overline{D}$. Since we cannot move it to the west, we have $(r-1, s) \in \widehat{D}$. It follows immediately that $r = k + a(k,s) + 1$ and $m = s + l(k,s)$. Therefore, $\phi : D \to A^0_{nw}$ is a bijection.

We illustrate this construction (and the definitions of arms and legs) on the Figure 1. ☐

One can repeat similar arguments for boxes $c \in \overline{D}$ in the complement to the diagram D. We will need to extend definitions arms and legs to the case of boxes in the complement:

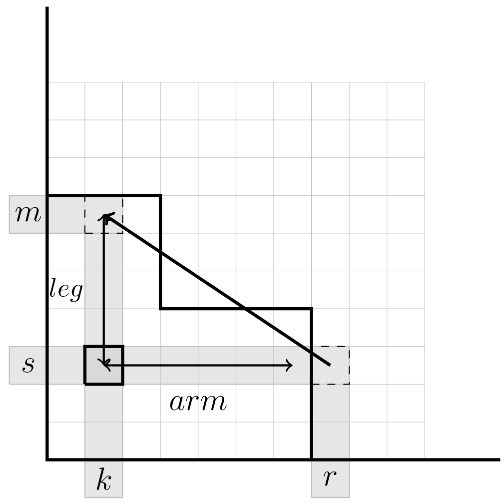

Figure 1: A northwest pointing arrow which cannot be moved north or west corresponds to a box inside the diagram D.

Definition 1.2. Let $c = (x, y) \in \overline{D}$ be a box in the complement $\overline{D}$ to a Young diagram D. The *arm* of the box c is the number of boxes outside D in the same row as c, to the west of c. Similarly, the *leg* of the box c is the number of boxes outside D in the same column as c, to the south of c. We use the notations $a(c) = a(x, y)$ and $l(c) = l(x, y)$ for the arm and the leg of the box c correspondingly.

The map $\overline{\phi} : \overline{D} \to A_{nw}$ defined as follows. Given a box $c = (x, y) \in \overline{D}$, we define

$$\overline{\phi}(x, y) = [(x, y - l(c)) \to (x - a(c) - 1, y)].$$

Theorem 1.2. *The map $\overline{\phi} : \overline{D} \to A_{nw}$ is a bijection.*

Proof. Let us now move the arrow to the south and to the east as much as possible. Suppose that the resulting arrow is $(r, s) \to (k, m)$. Since we cannot move it east anymore, we have $(k+1, m) \in \overline{D}$. Since we cannot move it south, we have $(r, s-1) \in \widehat{D}$. It follows immediately that we have $s = m - l(r, m)$ and $k = r - a(r, m) - 1$. Therefore, the map $\overline{\phi} : \overline{D} \to A_{nw}$ is a bijection.

We illustrate this construction on the Figure 2. $\square$

Note that for each integer vector $v = (-x, y)$, $x \in \mathbb{Z}_{>0}, y \in \mathbb{Z}_{\geq 0}$, there is exactly one class of escaping arrows of type v. Also, both in the case of

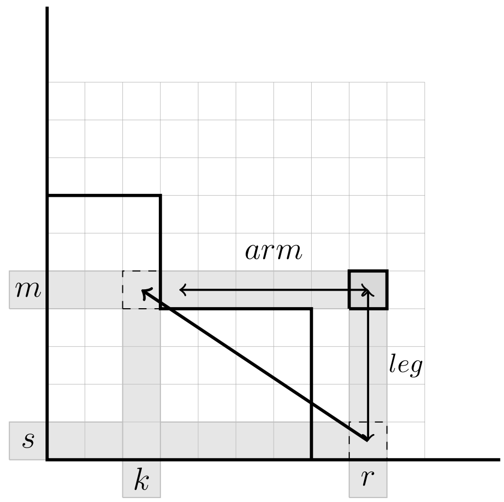

Figure 2: A northwest pointing arrow which cannot be moved south or east corresponds to a box outside the diagram D.

the Theorem 1.1 and in the case of the Theorem 1.2, the class of arrows corresponding to a box c is of type $(-a(c)-1, l(c))$. Therefore, we get the following corollary:

Corollary 1. *For any two fixed non-negative integers a and l, the number of boxes c inside the diagram D with $a(c) = a$ and $l(c) = l$ is one less than the number of boxes in the complement $\overline{D}$ with the same property.*

2 Inside a Rectangle.

Sometimes it is important to consider Young diagrams inscribed in a right triangle. Let p and q be positive integers, such that for all boxes $(x, y) \in D$ one has $qx + py \leq pq - p - q$ (remember that the southwest corner of D is $(0,0)$). Let $R_{p,q} := \{(x,y) \in (\mathbb{Z}_{\geq 0})^2 | x < p \text{ and } y < q\}$ denote the corresponding $p \times q$ rectangle, and $R^+_{p,q} := \{(x,y) \in (\mathbb{Z}_{\geq 0})^2 | qx + py \leq pq - p - q\}$ denote the set of boxes below the diagonal in it. We get $D \subset R^+_{p,q} \subset R_{p,q}$.

When modifying the results of the previous section to this new setup, one runs into an immediate problem: it might happen that the box $c \in \overline{D}$ corresponding to a class of arrows is outside the rectangle $R_{p,q}$. This might happen in two cases. First, the arrow might be not steep enough, so that as we move it to the east its tail moves outside the rectangle. And second, it

might be impossible to move an escaping arrow south enough for its head to be below the line $y = q$. Fortunately, both problems can be handled if one restricts his attention to "steep enough" arrows only.

Theorem 2.1. *Fix non-negative integers a and l such that $\frac{l}{a+1} \geq \frac{q}{p}$. Then one has two cases:*

1. *If $(a, q-1-l) \in D$, then the number of boxes c inside the diagram D such that $l(c) = l$ and $a(c) = a$ is equal to the number of boxes in the complement $R_{p,q} \backslash D$ with the same property.*

2. *If $(a, q-1-l) \notin D$, then the number of boxes c inside the diagram D such that $l(c) = l$ and $a(c) = a$ is one less then the number of boxes in the complement $R_{p,q} \backslash D$ with the same property.*

Proof. With the condition $\frac{l}{a+1} \geq \frac{q}{p}$ on the slope of arrows one cannot move a non-escaping arrow so that its tail is outside the triangle $R^+_{p,q}$. Indeed, otherwise its head is also above the diagonal, which contradicts the condition $D \subset R^+_{p,q}$. Therefore, the only class of arrows that might not be represented by a box in the complement $R_{p,q} \backslash D$ is the escaping class.

Now, if $(a, q-1-l) \notin D$ then the arrow $(a, q-1-l) \to (-1, q-1)$ belongs to the escaping class. Easy to see that in this case the box $c \in \overline{D}$ representing the escaping class is inside the rectangle $R_{p,q}$. Otherwise, the box is outside the rectangle. Indeed, if the box is inside then one should be able to move the arrow so that its head is at $(-1, q-1)$. We illustrate the proof on the Figure 3. $\square$

Applying Theorem 2.1 to all couples of numbers a and l satisfying the condition $\frac{l}{a+1} \geq \frac{q}{p}$ one gets the following corollary:

Corollary 2. *Number of boxes c inside D such that $\frac{l(c)}{a(c)+1} \geq \frac{q}{p}$ plus number of boxes in $R^+_{p,q} \backslash D$ is equal to the number of boxes c' in $R_{p,q} \backslash D$ such that $\frac{l(c')}{a(c')+1} \geq \frac{q}{p}$.*

In our joint paper with Eugene Gorsky [3], we proved this corollary by constructing an explicit bijection in the case when p and q are coprime.

Note that using the southeast pointing arrows instead of northwest, one obtains a similar result about boxes $c \in D$ satisfying $\frac{a(c)}{l(c)+1} \geq \frac{p}{q}$ or, equivalently, $\frac{l(c)+1}{a(c)} \leq \frac{q}{p}$:

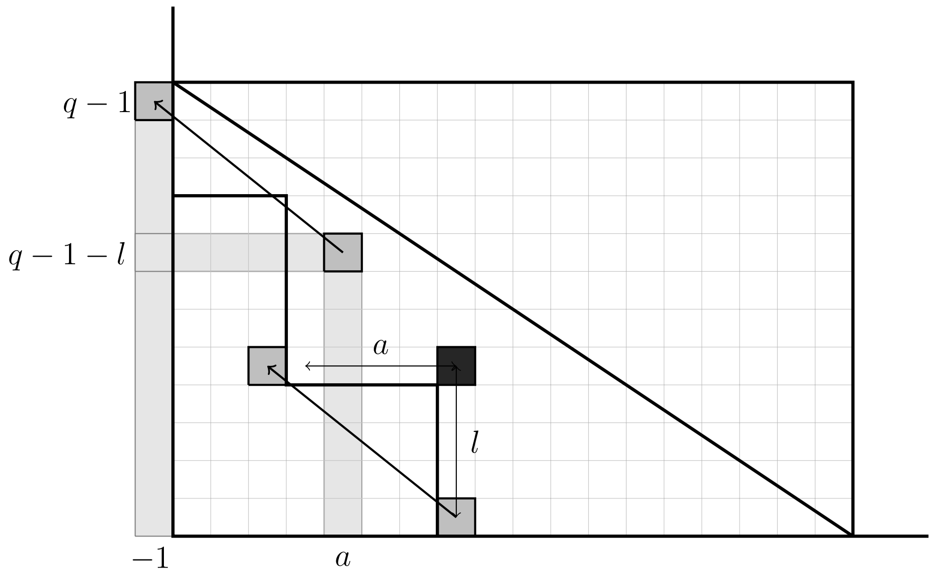

Figure 3: Two arrows representing the same escaping class. The top one is $(a, q-1-l) \to (-1, q-1)$, and the bottom one cannot be moved south or east. The class corresponds to the dark gray box in the complement $R_{p,q} \backslash D$.

Theorem 2.2. *Fix non-negative integers a and l such that $\frac{l+1}{a} \leq \frac{q}{p}$. Then one has two cases:*

1. *If $(p-1-a, l) \in D$, then the number of boxes c inside the diagram D such that $l(c) = l$ and $a(c) = a$ is equal to the number of boxes in the complement $R_{p,q} \backslash D$ with the same property.*

2. *If $(p-1-a, l) \notin D$, then the number of boxes c inside the diagram D such that $l(c) = l$ and $a(c) = a$ is one less then the number of boxes in the complement $R_{p,q} \backslash D$ with the same property.*

Proof. The same as for Theorem 2.1 with "north" switched with "south", and "west" switched with "east". □

3 Loehr-Warrington's identities.

In [5] the following statistics were considered:

Definition 3.1. [5] Let D be a Young diagram. Let

$$c^+_{p,q}(D) = \left| \left\{ c \in D : \frac{l(c)}{a(c)+1} = \frac{q}{p} \right\} \right|,$$

$$c^-_{p,q}(D) = \left| \left\{ c \in D : \frac{l(c)+1}{a(c)} = \frac{q}{p} \right\} \right|,$$

$$\mathrm{ctot}_{p,q}(D) = c^+_{p,q}(D) + c^-_{p,q}(D),$$

and

$$\mathrm{mid}_{p,q}(D) = \left| \left\{ c \in D : \frac{l(c)}{a(c)+1} < \frac{q}{p} < \frac{l(c)+1}{a(c)} \right\} \right|.$$

We would like to deduce Loehr-Warrington's formulas for $\mathrm{ctot}_{p,q}(D)$ and $\mathrm{mid}_{p,q}(D)$, using the Theorem 2.1. We will need the following construction from [5]. Consider the boundary lattice path $P(D)$ going from the southeast corner of the rectangle $R_{p,q}$ to the northwest corner of the rectangle $R_{p,q}$ along the boundary of the diagram D. We think of $P(D)$ as of an oriented graph with edges labeled by N (northward) and E (eastward). Let us label the vertices of $P(D)$ by integers as follows: the starting vertex is labeled by 0 and then each eastward edge adds q, while each northward edge subtracts p. Finally, we glue together the vertices labeled by the same integer. The resulting graph is denoted $M(D)$. Note that the graph $M(D)$ comes equipped with an Eulerian tour $E(D)$, following the path $P(D)$. We illustrate this construction on Figure 4.

Note that the boxes of the rectangle $R_{p,q}$ are in natural one-to-one correspondence with couples of edges of $M(D)$, one northward, and one eastward. Indeed, every row of $R_{p,q}$ contains exactly one northward edge of $P(D)$, and every column contains exactly one eastward edge. Moreover, boxes inside D correspond to couples for which the northward edge goes before the eastward in the Eulerian tour $E(D)$, and boxes outside D correspond to couples for which the eastward edge goes first.

Following Loehr and Warrington we introduce the following notations. Let V_M be the set of vertices of $M = M(D)$. We identify the vertices of M with the corresponding integers, so that $V_M \subset \mathbb{Z}$. For a vertex $v \in V_M$ let $E_{in}(v)$ be the set of eastward edges entering v. Respectively, let $N_{in}(v)$ be the set of northward edges entering v. The following Lemma follows immediately from the definitions:

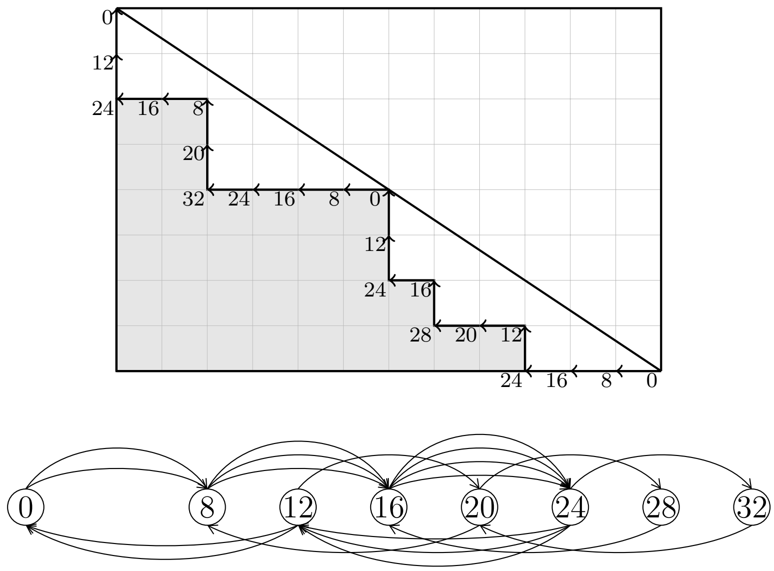

Figure 4: Example of a boundary path $P(D)$ and the corresponding graph $M(D)$. Here $p = 12$ and $q = 8$.

Lemma 3.1. *Let $c \in D$. Suppose that $e_c \in E_{in}(v)$ is the eastward edge corresponding to c, and $n_c \in N_{in}(w)$ is the northward edge corresponding to c. Then $v = w + (a(c)+1)q - l(c)p$. In particular,*

1. *One has $v = w$ if and only if $\frac{l(c)}{a(c)+1} = \frac{q}{p}$,*

2. *One has $v < w$ if and only if $\frac{l(c)}{a(c)+1} > \frac{q}{p}$.*

Similarly, if $c \in R_{p,q} \backslash D$, $e_c \in E_{in}(v)$, and $n_c \in N_{in}(w)$, then $w = v + a(c)q - (l(c)+1)p$. In particular,

1. *One has $v = w$ if and only if $\frac{l(c)+1}{a(c)} = \frac{q}{p}$,*

2. *One has $v < w$ if and only if $\frac{l(c)+1}{a(c)} < \frac{q}{p}$.*

Proof. The proof is immediate from the definitions. We illustrate it on Figure 5. □

Finally, we use Lemma 3.1 and Theorem 2.1 to deduce Loehr-Warringtons formulas. Let us start with $\mathrm{ctot}_{p,q}$:

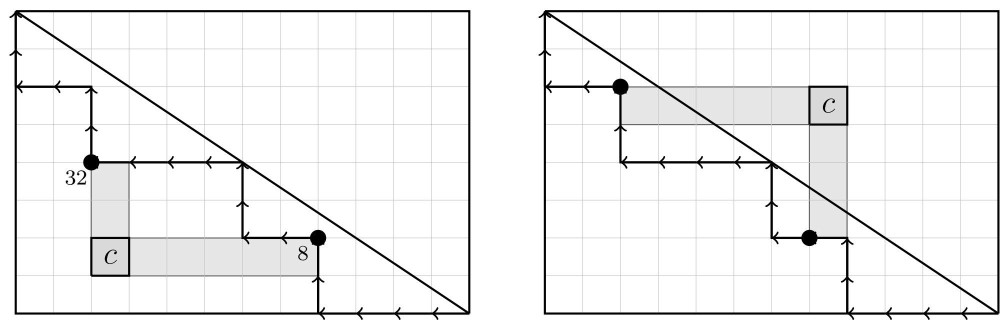

Figure 5: On this picture we have $p = 12$, $q = 8$. On the left we have a box $c \in D$ with $a(c) = 5$ and $l(c) = 2$. The corresponding northward arrow belongs to $N_{in}(8)$, and the corresponding eastward arrow belongs to $E_{in}(32)$, where $32 = 8 + 6 \times 8 - 2 \times 12$, because there are $6 = a(c) + 1$ eastward and $2 = l(c)$ northward arrows between the corresponding vertices. Similarly, on right we have $c \notin D$ with $a(c) = 5$ and $l(c) = 3$. We see that there are $a(c) = 5$ eastward and $l(c) + 1 = 4$ northward arrows between the corresponding vertices.

$$c^{+}_{p,q}(D) = \left| \left\{ c \in D : \frac{l(c)}{a(c)+1} = \frac{q}{p} \right\} \right| =$$

$$= \left| \left\{ c \in R_{p,q} \backslash D : \frac{l(c)}{a(c)+1} = \frac{q}{p} \right\} \right| - \left| \{(x,y) \notin D : qx + py = pq - p - q\} \right|.$$

Indeed, points $(x, y) \notin D$ such that $qx + py = qp - p - q$ are exactly those for which the arrow $(x, y) \to (-1, q - 1)$ has the required slope $\frac{q}{p}$. We conclude:

$$\mathrm{ctot}_{p,q}(D) = c^{-}_{p,q}(D) + c^{+}_{p,q}(D) =$$

$$= \sum_{v \in V_M} |E_{in}(v)||N_{in}(v)| - |\{(x,y) \notin D : qx + py = pq - p - q\}| =$$

$$= \sum_{v \in V_M} |E_{in}(v)||N_{in}(v)| - \gcd(p,q) + |N_{in}(0)|.$$

The last equality is justified as follows: there are exactly $\gcd(p, q) - 1$ boxes (x, y) in $R_{p,q}$ such that $qx+py = pq-p-q$, and $|N_{in}(0)|-1$ such boxes inside D. Indeed, all such boxes correspond to vertices of the boundary path labeled by 0, and we always arrive at such vertices along northward edges. Finally, we subtract one for the initial vertex of the path.

Statistic $\mathrm{mid}_{p,q}(D)$ can be treated similarly, using Theorem 2.2:

$$\begin{aligned}
\mathrm{mid}_{p,q}(D) = & \left|\left\{c \in D : \frac{l(c)}{a(c)+1} < \frac{q}{p} < \frac{l(c)+1}{a(c)}\right\}\right| = \\
= |D| - & \left|\left\{c \in D : \frac{l(c)}{a(c)+1} \geq \frac{q}{p}\right\}\right| - \left|\left\{c \in D : \frac{q}{p} \geq \frac{l(c)+1}{a(c)}\right\}\right| = \\
= |D| - & \left|\left\{c \in D : \frac{l(c)}{a(c)+1} \geq \frac{q}{p}\right\}\right| - \left|\left\{c \in R_{p,q} \setminus D : \frac{q}{p} \geq \frac{l(c)+1}{a(c)}\right\}\right| + |R^+_{p,q} \setminus D| = \\
= |R^+_{p,q}| - & \sum_{v,w \in V_M, v \leq w} |E_{in}(v)||N_{in}(w)|.
\end{aligned}$$

These formulas were first proved by Loehr and Warrington in [5] by induction.

4 Remarks on Geometry.

Statistics $\mathrm{ctot}_{p,q}$, $c^-_{p,q}$, $c^+_{p,q}$, and $\mathrm{mid}_{p,q}$ have nice geometric interpretations in terms of the toric action on the Hilbert scheme of points on the complex plane. The Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ is the space of ideals of codimension n in the polynomial ring $\mathbb{C}[x, y]$. It inherits a natural action of the two-dimensional torus $(\mathbb{C}^*)^2$, acting by scaling on the variables x and y. Fixed points are the monomial ideals, naturally parametrized by Young diagrams: given a Young diagram D, the corresponding monomial ideal I_D is spanned by monomials $x^k y^m$ for $(k, m) \notin D$.

Consider a one-dimensional subtorus $T_{p,q} = \{t^p, t^q\} \subset (\mathbb{C}^*)^2$, where p and q are positive coprime integers. If $p + q \leq n$, then the fixed points of the $T_{p,q}$ action are not isolated. The fixed point sets are called *quasihomogeneous Hilbert schemes* and denoted $\mathrm{Hilb}^n_{p,q}(\mathbb{C}^2)$. They are smooth and compact, but might be reducible and, moreover, irreducible components might have different dimensions.

Irreducible components of $\mathrm{Hilb}^n_{p,q}(\mathbb{C}^2)$ were studied by Evain ([2]). One can reformulate Evain's results to show that two monomial ideals belong to the same irreducible component of $\mathrm{Hilb}^n_{p,q}(\mathbb{C}^2)$ if and only if the corresponding Young diagrams share the same graph $M(D)$. Furthermore, one can use results of Ellingsrud and Stromme [1] to show that $\mathrm{ctot}_{p,q}(D)$ is equal to the dimension of the irreducible component of the quasihomogeneous Hilbert

scheme containing the ideal I_D, and $\mathrm{mid}_{p,q}(D)$ is the dimension of the unstable variety of I_D under the action of $T_{p,q}$. That provides a geometric explanation of the fact that $\mathrm{ctot}_{p,q}(D)$ and $\mathrm{mid}_{p,q}(D)$ depend only on the graph $M(D)$ and not on the Eulerian tour $E(D)$.

Moreover, the factor torus $T^{p,q} = (\mathbb{C}^2)/T_{p,q}$ acts on $\mathrm{Hilb}^n_{p,q}(\mathbb{C}^2)$ with isolated fixed points (monomial ideals), which gives rise to two Białynicki-Birula cell decompositions of $\mathrm{Hilb}^n_{p,q}(\mathbb{C}^2)$: into stable and into unstable varieties of fixed points (see [6]). One can show that $c^-_{p,q}(D)$ is the dimension of the stable variety of the fixed point $I_D \in \mathrm{Hilb}^n_{p,q}(\mathbb{C}^2)$, and $c^+_{p,q}(D)$ is the dimension of the unstable variety. Then, existence of the Loehr-Warrington's bijections, interchanging statistics $c^-_{p,q}$ and $c^+_{p,q}$ while preserving the multigraph $M(D)$, follows from the fact that both cell decompositions should have the same number of cells of a given dimension. However, geometric meaning of a particular bijection constructed in [5] remains mysterious.

References

[1] G. Ellingsrud and S. A. Stromme. On the homology of the Hilbert scheme of points in the plane. *Invent. Math.*, 87:343–352, 1987.

[2] L. Evain. Irreducible components of the equivariant punctual Hilbert schemes. *Adv. Math.*, 185(2):328–346, 2004.

[3] E. Gorsky and M. Mazin. Compactified Jacobians and q,t-Catalan Numbers, I. *Journal of Combinatorial Theory, Series A*, 120:49–63, 2013.

[4] M. Haiman. t, q-Catalan numbers and the Hilbert scheme. *Discrete Math.*, 193(1–3):201–224, 1998.

[5] N. Loehr and G. Warrington. A continuous family of partition statistics equidistributed with length. *Journal of Combinatorial Theory, Series A*, 116:379–403, 2009.

[6] A. Białynicki Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astron. Phys.*, 24:667–674, 1976.